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The evolution of unstable black holes in anti-de Sitter space

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Abstract: We examine the thermodynamic stability of large black holes in four-dimensional anti-de Sitter space, and we demonstrate numerically that black holes which lack local thermodynamic stability often also lack stability against small perturbations. This shows that no-hair theorems do not apply in anti-de Sitter space. A heuristic argument, based on thermodynamics only, suggests that if there are any violations of Cosmic Censorship in the evolution of unstable black holes in anti-de Sitter space, they are beyond the reach of a perturbative analysis.

Keywords: D-branes, Black Holes in String Theory, AdS-CFT Correspondance.

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1. Introduction

The Gregory-Laflamme instability [1] is a classical instability of black brane solutions in which the mass tends to clump together non-uniformly. The intuitive explanation for this instability is that the entropy of an array of black holes is higher for a given mass than the entropy of the uniform black brane. The intuitive explanation leaves something to be desired, since it applies equally to near-extremal Dp-branes: scaling arguments establish that a sparse array of large black holes threaded by an extremal Dp-brane will be entropically favored over a uniform non-extremal Dp-brane; however it is not expected that near-extremal Dp-branes exhibit the type of instability found in [1]. It was checked in [2] that a Dp-brane which is far from extremality (that is, one whose tension is many times the extremal tension) does have an instability. Less is known about the near-extremal case, but one may take the absence of tachyons in the extensive AdS-glueball calculations ([3, 4] and subsequent works — see [5] for a review) as provisional evidence that the near-extremal D3-brane, M2-brane, and M5-brane are (locally) stable.[1]

[1] More properly, we should say that the near-extremal black brane solutions with many units of D3-brane, M2-brane, or M5-brane charge appear to be stable. A single brane has Planck scale curvatures near the horizon, so classical two-derivative gravity does not provide a reliable description. We will concern ourselves exclusively with solutions which have a discrete parameter (M2-brane charge, for the most part) which can be dialed to infinity to suppress all corrections to classical gravity.

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In the AdS/CFT correspondence [6, 7, 8], one might at first think that the existence of a unitary field theory dual forbids an instability. But suppose we are at finite temperature, and that there is a thermodynamic instability in the field theory — like the onset of a phase transition. Then it is quite natural for some fluctuation mode (or modes) to grow exponentially in time, at least in a linearized analysis, as one nucleates the new phase. Exciting an unstable mode is a change in the state of the field theory, not its lagrangian; thus according to AdS/CFT there should be a *normalizable* mode in AdS which likewise grows exponentially with time [9]. This might be referred to as a "boundary tachyon," or a "tachyonic glueball," since in the gauge theory it corresponds to some bound state with negative mass-squared. We will prefer the term "dynamical instability," which is meant to convey that there is an instability in the lorentzian time evolution of the black brane, in both its supergravity and dual field theory descriptions.

To sum up, the existence of a field theory dual makes plausible the following adaptation of the entropic justification for the Gregory-Laflamme instability:

For a black brane solution to be free of dynamical instabilities, it is necessary and sufficient for it to be locally thermodynamically stable. (1.1)

Here, local thermodynamic stability is defined as having an entropy which is concave down as a function of the mass and the conserved charges. This criterion was first used in a black brane context in [10], where it was found that spinning D3-branes could be made locally thermodynamically unstable if the ratio of the spin to the entropy was high enough. Further work in this direction, relevant to the current paper, has appeared in [11]–[14]. For a somewhat complementary point of view on the nature of the unstable solutions, see [15, 16].

The conjecture (1.1) is meant to be a local version of the argument about whether the array of black holes or the black brane has higher entropy; however it seems on more precarious ground since one may not be able to write down a non-uniform stationary solution that competes with the black brane entropically. Nonetheless, it was shown in [17] that (1.1) predicts with good accuracy the value of the charge where the four-dimensional anti-de Sitter Reissner-Nordstrom solution (AdS_4-RN) develops an instability.

The aim of this paper is to give a fuller exposition of the calculations in [17], to present a more complete picture of the results of the numerics, and to explore via thermodynamic arguments the likely paths for time-evolution of the unstable solutions. Section 2 contains a summary of the AdS_4-RN solution and some generalizations of it in $\mathcal{N} = 8$ gauged supergravity and in higher dimensions. Section 3 discusses the thermodynamic instability which occurs for large charge. In section 4, a linear perturbation analysis is carried out around the AdS_4-RN solution. In the large black hole limit, a dynamical instability appears when local thermodynamic stability is lost. The existence of a dynamical instability was the main result of [17]. It disproves the claim of [18, 19] that charged black holes in AdS are classically stable. As we explain in section 4, the instability persists some ways away from the large black hole limit, providing the first proven example of a black hole with a compact horizon and a pointlike singularity which exhibits a dynamical Gregory-Laflamme instability.[2] Such solutions are interesting from the point of view of Cosmic Censorship, and we discuss the possibility of forming a naked singularity, or at least regions of arbitrarily large curvatures. Our main result here is that adiabatic evolution toward maximum entropy does not lead to solutions which arise from making the mass smaller than some appropriate combination of the charges. Because entropic arguments appear to give good information not only on the existence of dynamical instabilities but also on the direction they point, it is reasonable to predict from our results that no perturbative analysis of a smooth black hole in AdS will demonstrate a violation of Cosmic Censorship.

The unstable mode of the AdS_4-RN solution does not involve fluctuations of metric at linear order. Rather, it involves the gauge fields and scalars of $\mathcal{N} = 8$ gauged supergravity. Because the metric is not fluctuating, it may seem odd to describe the process as a Gregory-Laflamme instability. But we claim that the instability we see is in the same "universality class" as instabilities where the horizon does fluctuate: to be more precise, if the charges of the black hole are made slightly unequal, then generically the instability will involve the metric. In fact, the metric does fluctuate in the equal charge case as well — only at a subleading order that is beyond the scope of our linearized perturbation analysis. We would in fact make the case that any dynamical instability of a black hole which leads to non-uniformities in charge or mass densities should be considered in the same category as the Gregory-Laflamme instability of uncharged black branes.

We emphasize that this paper is concerned with the relation between local thermodynamic stability of stationary solutions and the stability of their classical evolution in lorentzian time. It is known [21]–[26] that black holes which are thermodynamically unstable have an unstable mode in the euclidean time formalism. For spherically symmetric black holes this mode is an s-wave. The interpretation is that, for instance, an AdS-Schwarzschild black hole in contact with a thermal bath of radiation will not equilibrate with the bath if the specific heat of the black hole is negative. This beautiful story does not fall under the rubric of problems we are considering, because the processes by which equilibration takes place in lorentzian time include Hawking radiation, which is non-classical. Rather, we are contemplat-

[2]Here we are referring to the existence of a local instability visible in a classical analysis. It has been observed [20] that the AdS-Schwarzschild solution times a sphere can have a lower entropy than a Schwarzschild black hole of the same mass which is localized on the sphere. This demonstrates global but not local instability, and suggests the possibility of tunneling from one configuration to the other.

ing black holes or branes in isolation from other matter, in a classical limit where Hawking radiation is suppressed, and inquiring whether a stationary, uniform black object wants to stay uniform or get lumpy as lorentzian time passes. It is less clear that there should be any relation between this dynamical question and local thermodynamic stability: for instance, a Schwarzschild black hole in asymptotically flat space is stable.[3] Yet we conjecture that (1.1) gives a precise relation when the black object has a non-compact translational symmetry.

Our focus in this paper is black holes in AdS and their black brane limits; however the conjecture (1.1) is intended to apply equally to any black brane. It may apply even beyond the regime of validity of classical gravity. Any "sensible" gravitational dynamics should satisfy the Second Law of Thermodynamics, and (1.1) is motivated solely by intuition that lorentzian time evolution should proceed so as to increase the entropy. (The stipulation of translational invariance prevents finite volume effects from vitiating simple thermodynamic arguments). For instance, it has recently been shown [29] that the near-extremal NS5-brane has a negative specific heat arising from genus one contributions on the string worldsheet (see also [30, 31], and [32] for related phenomena in 1+1-dimensional string theory).[4] This is not classical gravity, but (1.1) leads us to expect an instability in the lorentzian time evolution of near-extremal NS5-branes.[5] The instability would drive the NS5-brane to a state in which the energy density is non-uniformly distributed over the world-volume.

2. The AdS_4-RN solution and its cousins

The bosonic part of the lagrangian for $\mathcal{N} = 8$ gauged supergravity [33, 34] in four dimensions involves the graviton, 28 gauge bosons in the adjoint of SO(8), and 70 real scalars. Because of the scalar potential introduced by the gauging procedure, flat Minkowski space is not a vacuum solution of the theory; rather, AdS_4 is. It is known [35] that the maximally supersymmetric AdS_4 vacuum of $\mathcal{N} = 8$ gauged supergravity represents a consistent truncation of 11-dimensional supergravity compactified on S^7. The $AdS_4 \times S^7$ solution can be obtained as the analytic completion of the near-horizon limit of a large number of coincident M2-branes.[6] Making the M2-branes near-extremal corresponds to changing AdS_4 to the AdS_4-Schwarzschild solution. Near-extremal M2-branes can also be given angular momentum in the eight transverse dimensions. There are four independent angular momenta, corresponding

[3]This stability is implied by classical no-hair theorems, see for example [27]. A more extensive list of references on no-hair theorems can be found in [28]. A consequence of the present work is that these theorems cannot be extended to charged black holes in AdS.

[4]We thank D. Kutasov for bringing [29, 32] to our attention.

[5]We thank M. Rangamani for a number of discussions on this point.

[6]As stated in the introduction, taking the number of M2-branes large makes the geometry smooth on the Planck/string scale and thus suppresses corrections to classical two-derivative gravity.

to the $\mathrm{U}(1)^4$ Cartan subgroup of SO(8): these reduce to electric charges in the AdS_4 description. The electrically charged black hole solutions can be obtained most efficiently by first making a consistent truncation of the full $\mathcal{N} = 8$ gauged supergravity theory to the $\mathrm{U}(1)^4$ gauge fields plus three real scalars. Consistent truncation means that any solution of the reduced theory can be embedded in the full theory, with no approximations. For our purposes, it can be viewed as a sophisticated technique for generating solutions. The truncated bosonic lagrangian is

$$\begin{aligned}
\mathcal{L} &= \frac{\sqrt{g}}{2\kappa^2}\left[R - \sum_{i=1}^{3}\left(\frac{1}{2}(\partial\varphi_i)^2 + \frac{2}{L^2}\cosh\varphi_i\right) - 2\sum_{A=1}^{4} e^{\alpha_A^i\varphi_i}(F_{\mu\nu}^{(A)})^2\right] \\
\text{where}\qquad \alpha_A^i &= \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.
\end{aligned} \tag{2.1}$$

We use the conventions of [36], in particular, the metric signature is $-+++$ and $G_4 = 1$. In [36] the electrically charged solutions were found to be

$$\begin{aligned}
ds^2 &= -\frac{F}{\sqrt{H}}dt^2 + \frac{\sqrt{H}}{F}dz^2 + \sqrt{H}z^2 d\Omega^2 \\
e^{2\varphi_1} &= \frac{h_1 h_2}{h_3 h_4} \quad e^{2\varphi_2} = \frac{h_1 h_3}{h_2 h_4} \quad e^{2\varphi_3} = \frac{h_1 h_4}{h_2 h_3} \\
F_{0z}^{(A)} &= \pm\frac{1}{\sqrt{8}h_A^2}\frac{Q_A}{z^2} \\
H &= \prod_{A=1}^{4} h_A \quad F = 1 - \frac{\mu}{z} + \frac{z^2}{L^2}H \quad h_A = 1 + \frac{q_A}{z} \\
Q_A &= \mu\cosh\beta_A\sinh\beta_A \quad q_A = \mu\sinh^2\beta_A\,,
\end{aligned} \tag{2.2}$$

where the signs on the gauge fields can be chosen independently. We will lose nothing by choosing them all to be $+$. The quantities Q_A are the physical conserved charges, and they correspond to the four independent angular momenta of M2-branes in eleven dimensions. The mass is [11]

$$M = \frac{\mu}{2} + \frac{1}{4}\sum_{A=1}^{4} q_A\,, \tag{2.3}$$

and the entropy is

$$S = \pi z_H^2\sqrt{H(z_H)}\,, \tag{2.4}$$

where z_H is the largest root of $F(z_H) = 0$. Only for sufficiently large μ do roots to this equation exist at all. When they don't, the solution is nakedly singular.

We will be most interested in the case where all four charges are equal, $q_A = q$. Then the solution can be written more conveniently in terms of a new radial variable,

$r = z + q$, and it takes the form

$$\begin{aligned} ds^2 &= -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \\ F_{0r} &= \frac{Q}{\sqrt{8}r^2} \\ f &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{L^2}\,, \end{aligned} \tag{2.5}$$

with the scalars set to 0. In (2.5), F_{0r} is the common value of all four gauge field strengths $F_{0r}^{(A)}$. The geometry (2.5) is a solution of pure Einstein-Maxwell theory with a cosmological constant: it is the AdS_4-RN solution.

There are related solutions to maximally supersymmetric gauged supergravity in five and seven dimensions, corresponding respectively to spinning D3-branes and spinning M5-branes. In the case of D3-branes, there are six transverse dimensions, the rotation group is SO(6), the Cartan subalgebra is $\mathrm{U}(1)^3$, and as a result there are three independent angular momenta (or charges in the Kaluza-Klein reduced description). In the case of M5-branes, there are five transverse dimensions, the rotation group is SO(5), the Cartan subalgebra is $\mathrm{U}(1)^2$, and there are two independent angular momenta/charges. We will record here only the Einstein frame metric in the Kaluza-Klein reduced description, in conventions where $G_N = 1$ and L is the radius of the asymptotic AdS space. For further information on these solutions, the reader is referred to [11, 15]. The metrics are

$$\begin{aligned} AdS_5: \quad & ds^2 = -H^{-\frac{2}{3}} F dt^2 + H^{\frac{1}{3}} \left(\frac{dr^2}{F} + r^2 d\Omega_3 \right) \\ & H = \prod_{A=1}^{3} h_A \quad F = 1 - \frac{\mu}{r^2} + \frac{r^2}{L^2} H \quad h_A = 1 + \frac{q_A}{r^2} \\ AdS_7: \quad & ds^2 = -H^{-\frac{4}{5}} F dt^2 + H^{\frac{1}{5}} \left(\frac{dr^2}{F} + r^2 d\Omega_3 \right) \\ & H = \prod_{A=1}^{2} h_A \quad F = 1 - \frac{\mu}{r^4} + \frac{r^2}{L^2} H \quad h_A = 1 + \frac{q_A}{r^4}\,. \end{aligned} \tag{2.6}$$

3. Thermodynamics

3.1 Generalities

Given the solutions (2.2) and (2.6), we may read off the entropy, the mass, and the conserved electric charges. Typically it is most straightforward to express these quantities in terms of the non-extremality parameter μ and the boost parameters β_A. However it is possible to eliminate μ and β_A and find a polynomial equation relating M, S, and the Q_A. This equation can be solved straightfowardly for M, but not in general for S. We will quote explicit results for $M = M(S, Q_1, \ldots, Q_n)$ in the next section. In this section we will discuss thermodynamic stability assuming that $M(S, Q_1, \ldots, Q_n)$ is known.

The microcanonical ensemble is usually specified by a function $S = S(M, Q_1, \ldots, Q_n)$. Assuming positive temperature (which is safe for regular black holes since the Hawking temperature can never be negative), one may always invert $M = M(S, Q_A)$ to $S = S(M, Q_A)$, where now we abbreviate $Q_1, \ldots, Q_n$ to Q_A. A standard claim in classical thermodynamics is that the entropy for "sensible" matter must be concave down as a function of the other extensive variables. Locally this means that the hessian matrix,

$$\mathbf{H}^S_{M,Q_A} \equiv \begin{pmatrix} \frac{\partial^2 S}{\partial M^2} & \frac{\partial^2 S}{\partial M \partial Q_B} \\ \frac{\partial^2 S}{\partial Q_A \partial M} & \frac{\partial^2 S}{\partial Q_A \partial Q_B} \end{pmatrix}, \tag{3.1}$$

satisfies $\mathbf{H}^S_{M,Q_A} \leq 0$, i.e. it has no positive eigenvalues. To understand what this requirement means, consider the simplest case where $n = 0$ and $\partial^2 S/\partial M^2 > 0$. This is the statement that the specific heat is negative. A substance with this property (in a non-gravitational setting, but equating mass with energy) is unstable: if we start at temperature T, then it is possible to raise the entropy without changing the total energy by having some regions at temperature $T + \delta T$ and others at $T - \delta T$. Since we are implicitly assuming a thermodynamic limit, it is irrelevant how big the domains of high and low temperature are. In a more refined description (e.g. Landau-Ginzburg theory), these domains might have a preferred size, or at least a minimal size.

In the more general setting of many independent thermodynamic variables, let us define intensive quantities

$$(y_0, y_1, \ldots, y_n) = (M/V, Q_1/V, \ldots, Q_n/V), \tag{3.2}$$

where V is the volume. Suppose that $\mathbf{H}^S_{M,Q_A}$ has a positive eigenvector: $\mathbf{H}^S_{M,Q_A}\vec{v} = \lambda\vec{v}$ with $\lambda > 0$. Through a variation

$$y_j \to y_j + \epsilon v_j, \tag{3.3}$$

where ϵ is a function of position which integrates to 0, we can raise the entropy without changing the total energy or the conserved charges. Thus positive eigenvectors of $\mathbf{H}^S_{M,Q_A}$ indicate the way in which mass density and charge density tend to clump. Presumably the eigenvector with the most positive eigenvalue gives the dominant effect.

The stability requirement $\mathbf{H}^S_{M,Q_A} \leq 0$ may be rephrased as $\mathbf{H}^M_{S,Q_A} \geq 0$, where $\mathbf{H}^M_{S,Q_A}$ is the hessian of M with respect to S and Q_A. This is easy to understand from a geometrical point of view. $\mathbf{H}^S_{M,Q_A} \leq 0$ says that all the principle curvatures of $S(M, Q_A)$ point toward negative S, or, equivalently, away from the point $(S, M, Q_A) = (\infty, 0, 0)$. Now, the point $(S, M, Q_A) = (0, \infty, 0)$ is on the opposite side of the co-dimension hypersurface defined by $S = S(M, Q_A)$ from

$(S, M, Q_A) = (0, \infty, 0)$. Thus all principle curvatures should point toward $(0, \infty, 0)$, which means that $\mathbf{H}^M_{S,Q_A} \geq 0$. To determine the region of thermodynamic stability we may thus require $\det \mathbf{H}^M_{S,Q_A} > 0$, and then take the smallest connected components around points which are known to be stable.

While regions of stability are conveniently calculated from $\mathbf{H}^M_{S,Q_A}$, it is not clear that the eigenvector of $\mathbf{H}^S_{M,Q_A}$ with the largest positive eigenvalue can be read off easily from $\mathbf{H}^M_{S,Q_A}$. So it is useful to express $\mathbf{H}^S_{M,Q_A}$ directly in terms of derivatives of $M(S, Q_A)$:

$$\begin{aligned}
\frac{\partial^2 S}{\partial M^2} &= -\frac{1}{(\partial M/\partial S)^3} \frac{\partial^2 M}{\partial S^2} \\
\frac{\partial^2 S}{\partial Q_A \partial M} &= \frac{1}{(\partial M/\partial S)^3} \left[-\frac{\partial M}{\partial S} \frac{\partial^2 M}{\partial Q_A \partial S} + \frac{\partial M}{\partial Q_A} \frac{\partial^2 M}{\partial S^2} \right] \\
\frac{\partial^2 S}{\partial Q_A \partial Q_B} &= \frac{1}{(\partial M/\partial S)^3} \left[-\left(\frac{\partial M}{\partial S}\right)^2 \frac{\partial^2 M}{\partial Q_A \partial Q_B} - \frac{\partial^2 M}{\partial S^2} \frac{\partial M}{\partial Q_A} \frac{\partial M}{\partial Q_B} \right. \\
&\qquad \left. + \frac{\partial M}{\partial S} \left(\frac{\partial M}{\partial Q_A} \frac{\partial^2 M}{\partial Q_B \partial S} + \frac{\partial M}{\partial Q_B} \frac{\partial^2 M}{\partial Q_A \partial S} \right) \right].
\end{aligned} \tag{3.4}$$

A prescription for dealing with energy functions which violate the convexity condition $\mathbf{H}^M_{S,Q_A} \leq 0$ is the Maxwell construction, where one replaces $M(S, Q_A)$ with its convex hull (or $S(M, Q_A)$ by its convex hull — it's the same thing). This formal procedure is equivalent to allowing mixed phases where some domains have higher mass density or charge density than others. The energy functions resulting from charged black holes in AdS have the curious property that the convex hull is completely flat in some directions, so that chemical potentials (after taking the convex hull) are everywhere zero. This arises because, in certain directions, M rises slower than any nontrivial linear function of the other extensive variables. In this situation the Maxwell construction does not make much sense, because the mixed phases that it calls for have charges and mass concentrated arbitrarily highly in a small region, while the rest of the "sample" is at very low charge and mass density. A similar example in the simpler context of no conserved charges would be a mass function $M(S)$ like the one in figure 1. Here the natural physical interpretation is that the region between A and B represents a stable phase, while the region to the right of B is unstable toward clumping most of its energy into small regions. This tendency would presumably be cut off by some minimal length scale of domains. The mass functions obtained from charged black holes in AdS look roughly like figure 1 along some slices of the space of possible (S, Q_A). The interpretation we will offer is that the black holes are stable in the regime of parameters where convexity holds, and that they become dynamically unstable toward clumping their charge and energy outside this region.[7]

[7]There is a subtlety, discussed in [12], about the precise location of the boundary between stable

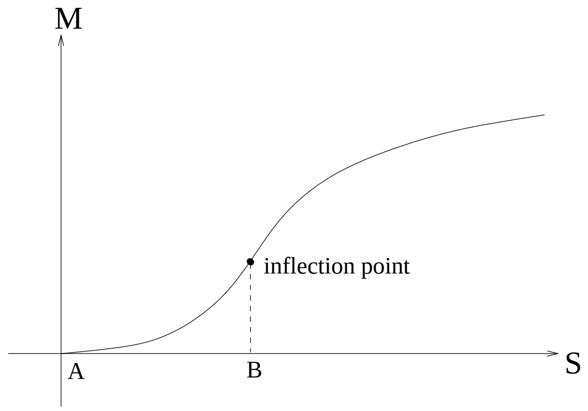

Figure 1: An example of a mass function whose convex hull is flat. The region we interpret as stable is from A to B.

The line of thought summarized in the previous paragraph was already advanced in [12], but with only thermodynamic arguments to support it. A competing point of view was suggested in [18]: the black holes in question have no ergosphere (more precisely, there is Killing vector field which is timelike everywhere outside the horizon), and this was argued to imply that there could be no superradiant modes, and hence no classical instability in the lorentzian-time dynamics. The argument used the dominant energy condition, which need not always be satisfied by matter in AdS: in fact, the scalars φ_i in (2.1) violate the dominant energy condition because of their tachyonic potential (which however does satisfy the Breitenlohner-Freedman bound).

In [17], an explicit numerical calculation demonstrated the existence of a dynamical instability for certain AdS_4-RN black holes (related to spinning M2-branes with all four spins equal, as explained in the previous section). We will discuss this calculation at greater length in section 4. For now let us only remark that in the limit of large black holes, where the horizon area is infinite, the instability appears when thermodynamic stability is lost, up to a discrepancy of 0.7% which we suspect is numerical error. Furthermore, the combination of supergravity fields which became unstable indicated a change in local charge densities precisely in agreement with the analysis leading up to (3.3). Thus the conjecture (1.1) was tested to reasonably good accuracy along a two-parameter subspace (entropy and the common value of the four charges) of the five-parameter phase space. Further tests in AdS_4 are significantly more difficult because the metric usually enters in to the perturbation equations in a non-trivial way. However we will indicate in section 4 another case where the metric decouples. Tests in AdS_5 and AdS_7 can also be performed most easily in the equal

and unstable regions. As the system approaches the inflection point at B, finite fluctuations might allow it to make small excursions into the unstable region. Working in a large N limit where classical supergravity applies on the AdS side of the duality seems to suppress such fluctuations.

charge case, but the analysis is somewhat more tedious because the spinor formalism is not as well worked out in higher dimensions (and probably is more cumbersome in any case).

Despite the absence of comprehensive tests, we will use (1.1) and the idea that black hole perturbations should follow the most unstable eigenvector of $\mathbf{H}^S_{M,Q_A}$ to propose in section 3.3 a qualitative picture of the evolution of unstable black holes in AdS. In brief, once the boundary of stability is passed, the independent charges tend to clump separately, as if they repelled one another but attracted themselves. But this is only an approximate tendency, with significant exceptions to be noted in section 3.3. When a particular angular momentum density becomes very large, it is possible that anti-de Sitter space fragments by a classical process.[8] This also will be discussed at greater length in section 3.3. We emphasize that our proposals for the evolution of unstable black holes are largely conjectural, and difficult to check by any means other than numerical solution of the full equations of motion.

3.2 Explicit formulas

It is possible to eliminate all the auxiliary quantities from (2.2), (2.3), and (2.4), and express M directly in terms of the entropy and the physical charges as

$$M = \frac{1}{2\pi^{\frac{3}{2}} L^2 \sqrt{S}} \left[\prod_{A=1}^{4} (S^2 + \pi L^2 S + \pi^2 L^2 {Q_A}^2) \right]^{1/4} . \tag{3.5}$$

We will often be interested in the limit of large black holes, $M/L \gg 1$. In this limit we have

$$M = \frac{1}{2\pi^{3/2} L^2 \sqrt{S}} \left[\prod_{A=1}^{4} (S^2 + \pi^2 L^2 {Q_A}^2) \right]^{1/4} . \tag{3.6}$$

with corrections suppressed by powers of M/L. As M/L approaches infinity, one obtains a black brane solution in the Poincare patch of AdS_4. Formally this limit can be taken by expanding (2.2) to leading order in small β_i, dropping the 1 from F, and replacing S^2 by R^2 in the metric.

As remarked in the previous section, local thermodynamic instability can be expressed as convexity of the function $M(S, Q_1, Q_2, Q_3, Q_4)$. By setting the hessian of (3.6) equal to zero, we obtain the boundary separating the stable from the unstable region:

$$3S^8 - 2\pi^2 L^2 S^6 \sum_{A=1}^{4} Q_A^2 + \pi^4 L^4 S^4 \sum_{A<B} (Q_A Q_B)^2 - \pi^8 L^8 \prod_{A=1}^{4} Q_A^2 = 0\,. \tag{3.7}$$

[8] Fragmentation of AdS via tunneling has been discussed in [37].

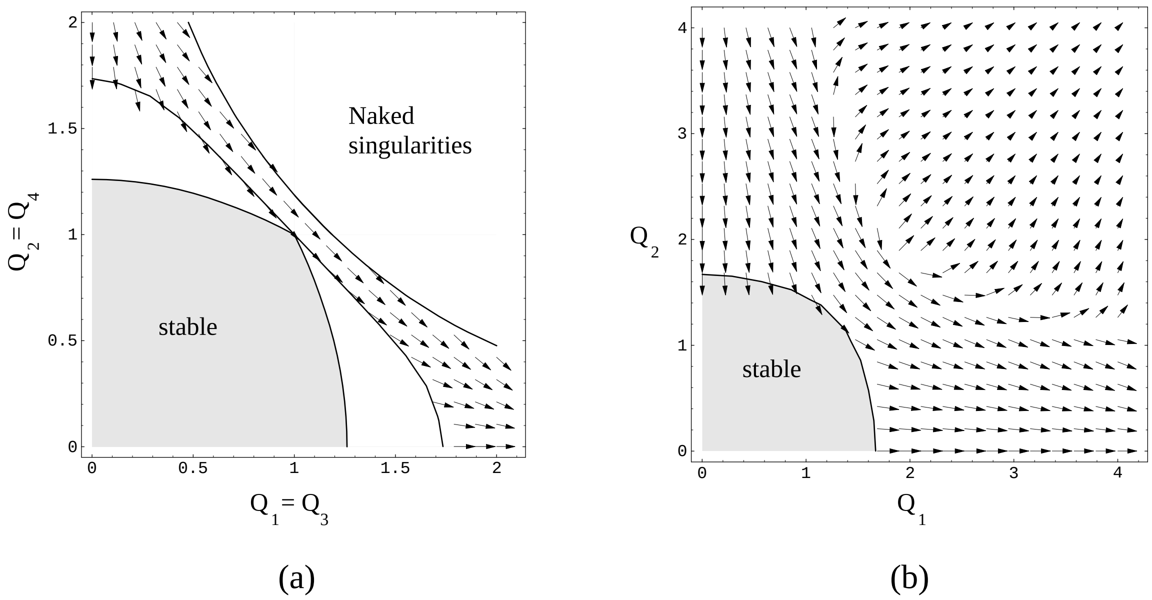

Figure 2: Plots of the most unstable eigenvector of the Hessian matrix of $S(M, Q_1, Q_2, Q_3, Q_4)$. The inner curves are boundaries of stability. The outer curves (when they are present) denote the boundary between regular black branes and naked singularities.

First let us consider the case where the charges are pairwise set equal: $Q_1 = Q_3$ and $Q_2 = Q_4$. The above equation then factorises, giving us three relevant factors:

$$\left(S^2 - \pi^2 L^2 Q_1^2\right)\left(S^2 - \pi^2 L^2 Q_2^2\right)\left(S^2 - \frac{\pi^2 L^2}{6}\left(Q_1^2 + Q_2^2 + \sqrt{Q_1^4 + Q_2^4 + 14 Q_1^2 Q_2^2}\right)\right) = 0\,. \tag{3.8}$$

When at least one of these factors become negative, $\mathbf{H}^M_{S,Q_A}$ develops a negative eigenvector and the black hole becomes thermodynamically unstable. A more convenient form may be obtained by eliminating S in favor of the mass M and introducing the dimensionless variable $\chi_i = \frac{Q_i}{M^{\frac{2}{3}} L^{\frac{1}{3}}}$. The above three equations in the new variables become

$$\begin{aligned}
\left[\chi_1^4 + \chi_2^4 + 8\chi_1^2\chi_2^2 + (\chi_1^2 + \chi_2^2)\sqrt{\chi_1^4 + \chi_2^4 + 14\chi_1^2\chi_2^2}\right]^2 - & \\
-54\left(\chi_1^2 + \chi_2^2 + \sqrt{\chi_1^4 + \chi_2^4 + 14\chi_1^2\chi_2^2}\right) &= 0 \\
\chi_1^6 + 2\chi_1^2\chi_2^2(\chi_1^2 + \chi_2^2) - 4 &= 0 \\
\chi_2^6 + 2\chi_1^2\chi_2^2(\chi_1^2 + \chi_2^2) - 4 &= 0\,.
\end{aligned} \tag{3.9}$$

The region depicting thermodynamically stable black holes is the intersection of the areas under the 3 curves as shown in figure $2(a)$.

The other relevant curve is the one separating nakedly singular solutions from regular black holes. The mathematical criterion for having a regular black hole solution is that the polynomial F in (2.2) should have a zero. In the large black hole limit, and in terms of χ_1 and χ_2, this criterion reduces to

$$\chi_1^2\chi_2^2(\chi_1^8+\chi_2^8) - 4\chi_1^4\chi_2^4(\chi_1^4+\chi_2^4) + 132\chi_1^2\chi_2^2(\chi_1^2+\chi_2^2) - 4(\chi_1^6+\chi_2^6) + \\ +6\chi_1^6\chi_2^6 - 432 = 0\,. \tag{3.10}$$

To determine if a black hole with given values of mass and charges is unstable, one first computes the values of χ_1 and χ_2 and locates this point in figure 2(a). The black hole is unstable if the point lies outside the shaded region depicting stable black holes but is within the boundary which separates black holes with naked singularities from those with a horizon. If the point lies in the unshaded (unstable) region of the plot without the vector field shown, it means that *within* each pair one charge wants to increase while the other decreases. The unstable eigenvector has no components along the hyperplane $Q_1 = Q_3$ and $Q_2 = Q_4$ and is not shown.

Finally, let us collect the thermodynamic results for the special case of all charges equal. We see that thermodynamic instability is present in the narrow region $1 < \chi < \sqrt{3}/2^{2/3}$. The associated eigenvector has the form $(0, 1, -1, 1, -1)$ where the components are along the axes M, Q_1, Q_2, Q_3, and Q_4 respectively: it looks like one pair of charges wants to increase while the other decreases. This can happen only locally, with each of the four charges conserved globally.

We'll also consider the case in which only two of the charges, Q_1 and Q_2, are non-zero. To get the region of thermodynamically stable black holes, we set $Q_3 = Q_4 = 0$ in (3.7):

$$3S^4 - 2\pi^2 L^2 S^2(Q_1^2+Q_2^2) + \pi^4 L^4 Q_1^2 Q_2^2 = 0\,. \tag{3.11}$$

Just as we did in the previous case, we first eliminate S in favor of the mass M and then introduce the dimensionless variables $\chi_i = \frac{Q_i}{M^{\frac{2}{3}} L^{\frac{1}{3}}}$ to get:

$$10(\chi_1^6+\chi_2^6) + 21\chi_1^2\chi_2^2(\chi_1^2+\chi_2^2) + \\ + \left(10\chi_1^4 + 10\chi_2^4 + 26\chi_1^2\chi_2^2\right)\sqrt{\chi_1^4+\chi_2^4-\chi_1^2\chi_2^2} - 432 = 0\,. \tag{3.12}$$

This is the boundary of the stable region, and is plotted in figure 2(b). Unlike the case of charges set equal pair-wise, black holes with two charges set to zero always have a horizon. This may be connected with the fact that there is a limit of rotating M2-branes with only two independent angular momenta nonzero which is a well-defined multi-center M2-brane solution, while with all angular momenta nonzero the corresponding limit is a singular configuration in eleven dimensions [38, 39, 40].

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For black holes in AdS_5 and AdS_7, we will simply record here the mass in terms of the entropy and charges:

$$\begin{aligned} AdS_5: \quad M &= \frac{3}{2L^2(2\pi^4 S)^{2/3}} \left[\prod_{A=1}^{3} (4S^2 + \pi^4 L^2 Q_A^2) \right]^{1/3} \\ AdS_7: \quad M &= \frac{5}{4L^2(4\pi^9 S)^{2/5}} \left[\prod_{A=1}^{2} (16S^2 + \pi^6 L^2 Q_A^2) \right]^{2/5} . \end{aligned} \tag{3.13}$$

Stability analyses similar to the AdS_4 case can be carried out for AdS_5 and AdS_7. Some work along these lines was presented in [12], but the explicit expressions in (3.13) make the calculations much easier.

3.3 Adiabatic evolution

Tracking the evolution of unstable black holes in lorentzian time is difficult. We have succeeded in establishing perturbatively the existence of a dynamical instability for the very special case of all charges equal: this is explained in section 4. This simplest case required the numerical solution of a fourth order ordinary differential equation with constraints at the horizon of the black hole and the boundary of AdS_4. Most other cases for black holes in AdS_4 involve fluctuations of the metric, which makes the analysis significantly harder. To investigate the instabilities beyond perturbation theory would require extensive numerical investigation of the second order PDE's that comprise the equations of motion of $\mathcal{N} = 8$ gauged supergravity.

The aim of this section is to use thermodynamic arguments to guess the qualitative features of the evolution of unstable black holes. Here we focus exclusively on the large black hole limit; however the conclusions may remain valid to an extent for finite size black holes with dynamical instabilities. The intuition is that knowing the entropy as a function of the other extensive parameters amounts to knowing the zero-derivative terms in an effective Landau-Ginzburg theory of the black hole (or of its dual field theory representation).

As explained in the paragraph around (3.2) and (3.3), an unstable eigenvector of $\mathbf{H}^S_{M,Q_A}$ (by which we mean one with positive eigenvalue) suggests a direction in which a black hole solution can be perturbed in order to raise entropy while keeping its total mass and conserved charges fixed; moreover it was shown in [17] (as we will explain in section 4) that the black hole's dynamical instability causes it to evolve in precisely the direction that the eigenvector indicates. The physics has no infrared cutoff, as is typical in Gregory-Laflamme setups, so we may hope that the charge and mass densities vary over long enough distance scales that we may continue to use the most unstable eigenvector of $\mathbf{H}^S_{M,Q_A}$ locally to determine the direction of the subsequent evolution. Following this line of thought to its logical conclusion leads us to the claim that the mass density and charge densities will locally evolve, subject to the constraints of conserving total energy and charge, from their initial values to values along a characteristic curve of the unstable vector field of $\mathbf{H}^S_{M,Q_A}$. This can only be approximately correct: finite wavelength distortions will occur, and it is not precisely right anyway to say that the time-evolution of Einstein's equations proceeds so as to maximize black hole entropy. Nevertheless it seems to us likely that a correct qualitative picture will emerge from tracking the flows generated by the most unstable eigenvector of $\mathbf{H}^S_{M,Q_A}$. At late times, or when charge and mass density are highly concentrated in small regions, another description is needed.

The characteristic curves of the most unstable eigenvector of $\mathbf{H}^S_{Q,M_A}$ may terminate in a region of stability, or in a region of naked singularities. Cosmic Censorship plus the conjectures of the previous paragraph suggest that the latter should never happen. This can be checked explicitly for the examples that we have. To this end, one can choose a generic value of charges and mass so that the black hole is almost naked, then determine the most unstable eigenvector of $\mathbf{H}^S_{M,Q_A}$, and then check that it is tangent to the surface separating naked singularities from regular black holes. We carried this out numerically for several cases and verified tangency; however we do not have a general argument. It appears, in fact, that the normal vector to the surface separating naked singularities from black holes is a stable eigenvector of $\mathbf{H}^S_{M,Q_A}$ (i.e. its eigenvalue is negative) — at least in the three-dimensional subspace with $Q_1 = Q_3$ and $Q_2 = Q_4$ — so the obvious approach to an analytic demonstration that Cosmic Censorship is not violated by adiabatic evolution of black holes is to show that this normal vector is always a stable eigenvector of $\mathbf{H}^S_{M,Q_A}$. For now we content ourselves with the observation that in all the cases we have checked numerically, adiabatic evolution does stay in the region of regular black holes.

It is also possible that a characteristic curve becomes unstable at some point, in the sense that nearby characteristic curves diverge from it. To refine our previous claim, we may suppose that the black hole evolves along a bundle of nearby characteristic curves emanating from the original mass and charge density. This bundle may remain nearly one-dimensional, or it may split or become higher dimensional. We will not investigate the stability properties of the characteristic curves in any detail. Note that we are not attempting to specify any spatial or temporal properties of the evolution, only the range of mass and charge densities which form.

We present in figure 2 plots of unstable eigenvectors of the hessian matrix $\mathbf{H}^S_{M,Q_A}$, projected onto a plane parametrized by two of the charges. From these vector fields, we may conclude that the different charges exhibit some tendency to separate from one another, but that this does not always happen, as in the upper right part of figure 2(b). The crucial point is that the unstable eigenvectors don't have a component normal to the boundary between naked singularities and regular black holes. Although this appears obvious from figure 2(a), the plot is slightly misleading in that the eigenvectors have been projected onto the plane of $Q_1 = Q_3$ and $Q_2 = Q_4$. One must preserve the components of vectors in the M direction to verify tangency.

When some angular momenta become large compared to the entropy for a spinning M2-brane solution, the geometry in eleven dimensions is approximately given by a rotating multi-center brane solution [38]. If one angular momentum is large, this multi-center solution is in the shape of a disk; if two are large and equal, it has the shape of a filled three-sphere. It seems clear that solutions of this form in an asymptotically flat eleven-dimensional spacetime are unstable toward fragmentation in the directions transverse to the M2-brane. This would mean that anti-de Sitter space fragments. In terms of the SU(N) gauge theory, the disk corresponds to a U$(1)^{N-1}$ Higgsing, and in the fragmentation process some groups of U(1)'s try to come together to partially restore gauge invariance. It is not certain that such fragmentation occurs, particularly if the angular momentum density is large only locally. We merely indicate it as a possibility in the complicated late-time evolution of unstable black holes.

Finally, it is possible that the unstable black holes evolve to a some new stationary solution, presumably with a non-uniform horizon. No such solutions are known. It seems to us most likely that, upon becoming unstable, black holes in AdS undergo an evolution which eventually produces large curvatures.

4. Existence of a dynamical instability

The existence of dynamical instabilities for thermodynamically unstable black branes should be completely generic. However, as mentioned already, the stability analysis is technically complicated for the general case of unequal charges: perturbations of the metric, four gauge fields, and three scalars lead to difficult coupled partial differential equations. Here we focus on the AdS_4-RN example, where the metric decouples and the problem can be reduced to a single gauge field and a single scalar. A formal argument relating thermodynamic and dynamical instability was suggested in [17], using the identification of the free energy with the euclidean supergravity action; however we have not yet succeeded in making this argument rigorous.

Because the unstable eigenvectors of $\mathbf{H}^S_{M,Q_A}$ (for all charges equal and sufficiently large) do not involve any change in the mass density, it is natural to expect that the perturbations that give rise to an unstable mode do not involve the metric.[9] More precisely, because of the form of the unstable eigenvectors, we expect that a relevant perturbation is

$$\delta F_A = \alpha^i_A \delta F \tag{4.1}$$

for some δF and fixed i, where the α^i_A were defined in (2.1). In section 3 we saw explicitly that $\delta Q_1 = \delta Q_3 = -\delta Q_2 = -\delta Q_4$ gave an unstable eigenvector; now we make a trivial alteration and focus on $\delta Q_1 = \delta Q_2 = -\delta Q_3 = -\delta Q_4$. Correspondingly we set $i = 1$ in (4.1).

[9]Indeed, we suspect that the decoupling of the metric is possible precisely when there is an eigenvalue of $\mathbf{H}^S_{M,Q_A}$ which does not have a component in the M direction.

The spectrum of linear perturbations to charged black holes in AdS has been considered before [41], but for the most part the perturbations under study were minimally coupled scalars. It is impractical to sift through the entire spectrum of supergravity looking for unstable modes (or tachyonic glueballs, in the language of [41]). The point of the previous paragraphs is that thermodynamics provides guidance not only on when to expect an instability, but also in which mode.

It is straightforward to start with the lagrangian in (2.1) and show that linearized perturbations to the equations of motion result in the following coupled equations:

$$\begin{gathered} d\delta F = 0 \qquad d * \delta F + d\delta\varphi_1 \wedge *F = 0 \\ \left[\Box + \frac{2}{L^2} - 8F_{\mu\nu}^2\right]\delta\varphi_1 - 16F^{\mu\nu}\delta F_{\mu\nu} = 0\,. \end{gathered} \tag{4.2}$$

Here $\Box = g^{\mu\nu}\nabla_\mu\partial_\nu$ is the usual scalar laplacian. F in (4.2) is the background field strength in (2.5): it is the common value of the four F_A. δF is *not* the variation in F itself; rather, the variation of the F_A is expressed in terms of δF in (4.1), with $i = 1$. The variation of the field strength is in a direction *orthogonal* to the background field strength of the AdS_4-RN solution. The graviton decouples from the linearized perturbation equations: $\delta T_{\mu\nu}$ vanishes at linear order in δF because $\delta F_A \cdot F_A = 0$.[10]

For comparison, we write down the linearized equations for fluctuations of the other scalars:

$$\left[\Box + \frac{2}{L^2} - 8F_{\mu\nu}^2\right]\delta\varphi_i = 0 \tag{4.3}$$

for $i = 2, 3$. It was shown in [18] that any perturbation involving only matter fields satisfying the dominant energy condition could not result in a normalizable unstable mode (that is, a normalizable mode which grows exponentially in lorentzian time). It was conjectured [18, 19] that in fact there was no classical instability at all. The scalars φ_i do not satisfy the dominant energy condition because of the potential term in (2.1). Thus the outcome of our calculations is not fore-ordained by general arguments, and we have a truly non-trivial check on the classical stability of highly charged black holes in $\mathcal{N} = 8$ gauged supergravity. In fact, our results turn out to be in conflict with the claim of classical stability in [18, 19].

Decoupling the equations in (4.2) is a chore greatly facilitated by the use of the dyadic index formalism introduced in [42]. For the reader interested in the details, we present an outline of the derivation in section 4.1. The final result is the fourth

[10] Besides the all-charges-equal case, we know of one other case where the metric decouples at linear order: $Q_1 = Q_3$ with $Q_2 = Q_4 = 0$. There may be other cases as well—presumably whenever $Q_A \cdot \delta Q_A = 0$ and $\delta S = 0$ for an unstable eigenvector $(\delta S, \delta Q_A)$ of $\mathbf{H}^M_{S,Q_A}$.

order ordinary differential equation (ODE)

$$\left(\frac{\omega^2}{f}+\partial_r f\partial_r-\frac{\ell(\ell+1)}{r^2}\right)r^3\left(\frac{\omega^2}{f}+\partial_r f\partial_r-\frac{\ell(\ell+1)}{r^2}-\frac{2M}{r^3}+\frac{4Q^2}{r^4}\right)r\delta\tilde{\varphi}_1(r)=$$
$$=4Q^2\left(\frac{\omega^2}{f}+\partial_r f\partial_r\right)\delta\tilde{\varphi}_1(r)\,, \tag{4.4}$$

where we have assumed the separated form $\delta\varphi_1=\Re e^{-i\omega t}Y_{\ell m}\delta\tilde{\varphi}_1(r)$, where $Y_{\ell m}$ is the usual spherical harmonic on S^2. This is to be compared with the separated equation for the other scalars:

$$\left(\frac{\omega^2}{f}+\partial_r f\partial_r-\frac{\ell(\ell+1)}{r^2}-\frac{2M}{r^3}+\frac{4Q^2}{r^4}\right)r\delta\tilde{\varphi}_i(r)=0 \tag{4.5}$$

for $i=2,3$.

4.1 Dyadic index derivation of (4.4)

To derive (4.4) using the dyadic index formalism, it is convenient first to switch to $+---$ signature to avoid sign incompatibilities between the raising and lowering of dyadic and vector indices. One introduces a null tetrad of vectors, $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, defined so that $l^\mu n_\mu = -m^\mu m_\mu = 1$ and all other inner products vanish. Next define

$$\sigma^\mu_{\Delta\dot{\Delta}}=\begin{pmatrix} l^\mu & m^\mu \\ \bar{m}^\mu & n^\mu \end{pmatrix} \tag{4.6}$$

and set $D=l^\mu\partial_\mu$, $\Delta=n^\mu\partial_\mu$, $\delta=m^\mu\partial_\mu$, $\bar{\delta}=\bar{m}^\mu\partial_\mu$. Vector indices are converted into dyadic indices by setting $v_{\Delta\dot{\Delta}}=\sigma^\mu_{\Delta\dot{\Delta}}v_\mu$. Dyadic indices are raised and lowered using northwest contraction rules with $\epsilon_{01}=\epsilon^{01}=\epsilon_{\dot{0}\dot{1}}=\epsilon^{\dot{0}\dot{1}}=1$. By demanding that $\sigma^\mu_{\Delta\dot{\Delta}}$ is covariantly constant, one can obtain a unique covariant derivative D_μ, whose action on a spinor is

$$D_\mu\psi_\Gamma=\partial_\mu\psi_\Gamma-\psi_\Sigma\gamma_\mu{}^\Sigma{}_\Gamma\,. \tag{4.7}$$

The so-called spin coefficients, $\gamma_{\Delta\dot{\Delta}\Sigma\Gamma}=\sigma^\mu_{\Delta\dot{\Delta}}\gamma_{\mu\,\Sigma\Gamma}$, are conventionally written as

$$\begin{aligned}\gamma_{0\dot{0}\Sigma\Gamma}&=\begin{pmatrix}\kappa & \epsilon\\ \epsilon & \pi\end{pmatrix} \qquad \gamma_{0\dot{1}\Sigma\Gamma}=\begin{pmatrix}\sigma & \beta\\ \beta & \mu\end{pmatrix}\\ \gamma_{1\dot{0}\Sigma\Gamma}&=\begin{pmatrix}\rho & \alpha\\ \alpha & \lambda\end{pmatrix} \qquad \gamma_{1\dot{1}\Sigma\Gamma}=\begin{pmatrix}\tau & \gamma\\ \gamma & \nu\end{pmatrix}\,.\end{aligned} \tag{4.8}$$

A less compressed presentation of dyadic index formalism can be found in [42, 43], and the appendix to [44].

For AdS_4-RN, a convenient choice of the null tetrad and the corresponding nonzero spin coefficients are as follows:

$$\begin{aligned} l^\mu &= \left(\frac{1}{f},1,0,0\right) & n^\mu &= \frac{1}{2}(1,-f,0,0)\\ m^\mu &= \frac{1}{r\sqrt{2}}(0,0,1,i\csc\theta) & \bar{m}^\mu &= \frac{1}{r\sqrt{2}}(0,0,1,-i\csc\theta)\end{aligned} \tag{4.9}$$

$$\rho = -\frac{1}{r} \quad \mu = -\frac{f}{2r} \quad \gamma = \frac{f'}{4} \quad \alpha = -\beta = -\frac{\cot\theta}{\sqrt{8}r}\,. \tag{4.10}$$

in 4.9 and 4.10 we have not yet taken the black brane limit. Taking this limit replaces $\csc\theta$ by 1 in 4.9 and sets $\alpha = \beta = 0$ in 4.10. Proceeding without the black brane limit, we trade the real antisymmetric tensor $F_{\mu\nu}$ for a complex symmetric tensor,

$$\Phi^{(0)}_{\Delta\Gamma} = \begin{pmatrix} \phi^{(0)}_0 & \phi^{(0)}_1 \\ \phi^{(0)}_1 & \phi^{(0)}_2 \end{pmatrix} \tag{4.11}$$

through the formula

$$4\sqrt{2}F_{\mu\nu}\sigma^{\mu}_{\Delta\dot{\Delta}}\sigma^{\nu}_{\Gamma\dot{\Gamma}} = \Phi^{(0)}_{\Delta\Gamma}\epsilon_{\dot{\Delta}\dot{\Gamma}} + \bar{\Phi}^{(0)}_{\dot{\Delta}\dot{\Gamma}}\epsilon_{\Delta\Gamma}\,. \tag{4.12}$$

The factor of $4\sqrt{2}$ in (4.11) is for convenience: the AdS_4-RN background has $\phi^{(0)}_1 = Q/r^2$ and all other components zero. In the same way we trade in $\delta F_{\mu\nu}$ for $\Phi_{\Delta\Gamma}$, whose components are ϕ_0, ϕ_1, and ϕ_2, with a similar factor of $4\sqrt{2}$. Finally, we write φ in place of $\delta\varphi_1$ to avoid the ambiguity in the meaning of δ.

The first order equations for the gauge field in (4.2) can now be cast in dyadic form as follows:

$$D^{\Delta}{}_{\dot{\Gamma}}\Phi_{\Delta\Gamma} + \frac{1}{2}\partial^{\Delta\dot{\Delta}}\varphi(\Phi^{(0)}_{\Delta\Gamma}\epsilon_{\dot{\Delta}\dot{\Gamma}} + \bar{\Phi}^{(0)}_{\dot{\Delta}\dot{\Gamma}}\epsilon_{\Delta\Gamma}) = 0\,. \tag{4.13}$$

In components, these equations read

$$\begin{aligned}
(D - 2\rho)\phi_1 - (\bar{\delta} - 2\alpha)\phi_0 &= -\phi^{(0)}_1 D\varphi \\
(\Delta + \mu - 2\gamma)\phi_0 - \delta\phi_1 &= 0 \\
(D - \rho)\phi_2 - \bar{\delta}\phi_1 &= 0 \\
(\delta + 2\beta)\phi_2 - (\Delta + 2\mu)\phi_1 &= \phi^{(0)}_1 \Delta\varphi\,.
\end{aligned} \tag{4.14}$$

It is possible to combine these equations into three second order equations in which only a single ϕ_i appears. Together with the scalar equation, these equations are equivalent to (4.2):

$$\begin{aligned}
\left[(D - 3\rho)(\Delta + \mu - 2\gamma) - \delta(\bar{\delta} - 2\alpha)\right]\phi_0 &= -\phi^{(0)}_1 \delta D\varphi \\
\left[(\Delta + 3\mu)(D - \rho) - \bar{\delta}(\delta + 2\beta)\right]\phi_2 &= -\phi^{(0)}_1 \bar{\delta}\Delta\varphi \\
\left[(D - 2\rho)(\Delta + 2\mu) - (\delta + \beta - \alpha)\bar{\delta}\right]\phi_1 &= -\phi^{(0)}_1 D\Delta\varphi \\
\left[\Box + \frac{2}{L^2} + 2(\phi^{(0)}_1)^2\right]\varphi &= -4\phi^{(0)}_1 \Re\phi_1\,,
\end{aligned} \tag{4.15}$$

where we have made use of the fact that the spin coefficients are all real for AdS_4-RN. The equations (4.14) are a special case of (3.1)-(3.4) of [45]. The first and second equations of (4.15) are (3.5) and (3.7) of [45], and the third is derived in a similar manner. The fourth is the scalar equation in (4.2), but to preserve the definition of

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$\Box$ we write $\Box = -g^{\mu\nu}\nabla_\mu\partial_\nu$ in $+---$ conventions. The differential operators in the third equation of (4.15) are purely real (this takes a bit of checking for $(\delta+\beta-\alpha)\bar{\delta}$), so we can take the real and imaginary parts of this equation. The equation for $\Im\phi_1$ decouples from all the others. The equations for ϕ_0 and ϕ_2 are sourced by φ, but ϕ_0 and ϕ_2 do not otherwise enter; thus one can solve first for $\Re\phi_1$ and φ, and afterwards use the first and second equations in (4.15) to obtain ϕ_0 and ϕ_2. Since $\phi_1^{(0)}$ is nowhere vanishing, the last equation in (4.15) can be used to eliminate $\Re\phi_1$ algebraically. The final result is

$$\left[(D-2\rho)(\Delta+2\mu)-(\delta+\beta-\alpha)\bar{\delta}\right]\frac{1}{4\phi_1^{(0)}}\left[\Box+\frac{2}{L^2}+2(\phi_1^{(0)})^2\right]\varphi=\phi_1^{(0)}D\Delta\varphi\,. \tag{4.16}$$

Plugging in the separated ansatz $\varphi=\Re\left\{e^{-i\omega t}Y_{\ell m}\delta\tilde{\varphi}_1(r)\right\}$, one easily obtains (4.4).

4.2 Numerical results from the fourth order equation

A dynamical instability exists if there is a normalizable, unstable solution to (4.4) or to (4.5). Neither of these equations admits a solution in closed form, so we have resorted to numerics. Briefly, the conclusion is that, in the black brane limit and within the limits of numerical accuracy, we find a single unstable mode for (4.4) precisely when $\chi>1$, and no instabilities for (4.5). This is completely in accord with the intuition from thermodynamics: (4.5) represents a fluctuation that has nothing to do with the variation of charges that gave the unstable eigenvector of the hessian matrix of $M(S,Q_1,Q_2,Q_3,Q_4)$. The unstable mode in (4.4) persists to finite size AdS_4-RN black holes, but eventually disappears for small enough black holes.

To carry out a numerical study of (4.4), the first step is to cast the equation in terms of a dimensionless radial variable u, a dimensionless charge parameter χ, a dimensionless mass parameter σ, and a dimensionless frequency $\tilde{\omega}$:

$$u=\frac{r}{M^{1/3}L^{2/3}}\qquad\chi=\frac{Q}{M^{2/3}L^{1/3}}\qquad\sigma=\left(\frac{L}{M}\right)^{2/3}\qquad\tilde{\omega}=\frac{\omega L^{4/3}}{M^{1/3}}\,. \tag{4.17}$$

Then we have

$$\begin{aligned}&\left(\frac{\tilde{\omega}^2}{\tilde{f}}+\partial_u\tilde{f}\partial_u-\sigma\frac{\ell(\ell+1)}{u^2}\right)u^3\left(\frac{\tilde{\omega}^2}{\tilde{f}}+\partial_u\tilde{f}\partial_u-\sigma\frac{\ell(\ell+1)}{u^2}-\frac{2}{u^3}+\frac{4\chi^2}{u^4}\right)u\delta\tilde{\varphi}_1=\\&=4\chi^2\left(\frac{\tilde{\omega}^2}{\tilde{f}}+\partial_u\tilde{f}\partial_u\right)\delta\tilde{\varphi}_1\\&f=\sigma-\frac{2}{u}+\frac{\chi^2}{u^2}+u^2\,.\end{aligned} \tag{4.18}$$

Evidently, the dimensionless control parameters are ℓ (the partial wave number), σ, and χ. Using Mathematica, we solved (4.18) numerically via a shooting method, and obtained wavefunctions $\delta\tilde{\varphi}_1(r)$ which fall off like $1/r^2$ near the boundary of AdS_4 and at least as fast as $(r-r_H)^{|\omega|/f'(r_H)}$ near the horizon.

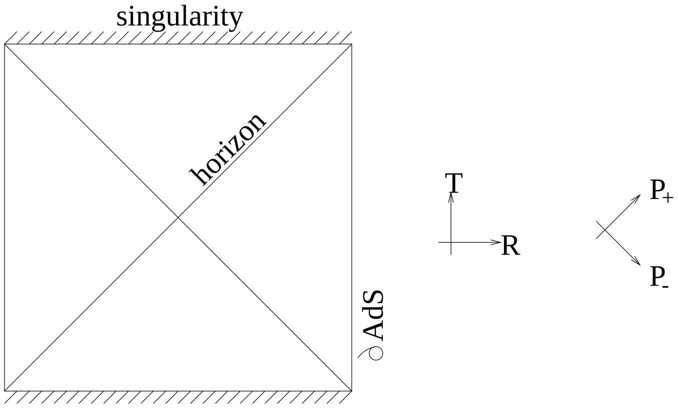

Figure 3: The Penrose diagram of a regular AdS black hole. We can take $T = R = P_+ = P_- = 0$ at the center of the diagram. The black hole horizon is the diagonal line going up and right from the origin.

To check that the wavefunction is well behaved near the horizon[11] let us transform to Kruskal coordinates. The metric near the horizon is

$$ds^2 \approx -f'(r_H)(r - r_H)dt^2 + \frac{dr^2}{f'(r_H)(r - r_H)} + r_H^2 d\Omega_2^2 \,, \tag{4.19}$$

where r_H is the radius of the horizon. Dropping the S^2 piece and introducing a tortoise coordinate r_*, null coordinates $P_\pm$, and Kruskal coordinates (T, R) according to

$$\begin{aligned} \frac{dr_*}{dr} &= \frac{1}{f'(r_H)(r - r_H)} \\ P_\pm &= e^{\frac{1}{2}f'(r_H)(\pm t + r_*)} = \pm T + R \,, \end{aligned} \tag{4.20}$$

one finds that the near-horizon metric is indeed regular:

$$ds_2^2 = -f'(r_H)(r - r_H)dt^2 + \frac{dr^2}{f'(r_H)(r - r_H)} = \frac{4}{f'(r_H)}(-dT^2 + dR^2) \,. \tag{4.21}$$

Having a radial wavefunction $\delta\tilde{\varphi}_1(r) = (r - r_H)^{|\omega|/f'(r_H)}\rho(r - r_H)$, where $\rho(r - r_H)$ remains bounded at the horizon, means that the time-dependent perturbation (with angular dependence suppressed) is

$$\delta\varphi_1(t, r) \sim (r - r_H)^{|\omega|/f'(r_H)} e^{|\omega| t} \rho(r - r_H) \sim P_+^{2|\omega|/f'(r_H)} \rho(P_+ P_-) \,, \tag{4.22}$$

which remains bounded as $P_- \to 0$. The black hole horizon is at $P_- = 0$, $P_+ > 0$ (see figure 3). Thus we see that the perturbation is small at the horizon in good coordinates, at least for small P_+. (As the perturbation grows, the horizon eventually starts to fluctuate, but this is not an issue in the question of whether the instability exists).

[11] We thank G. Horowitz for suggesting that this check should be made.

a)

b)

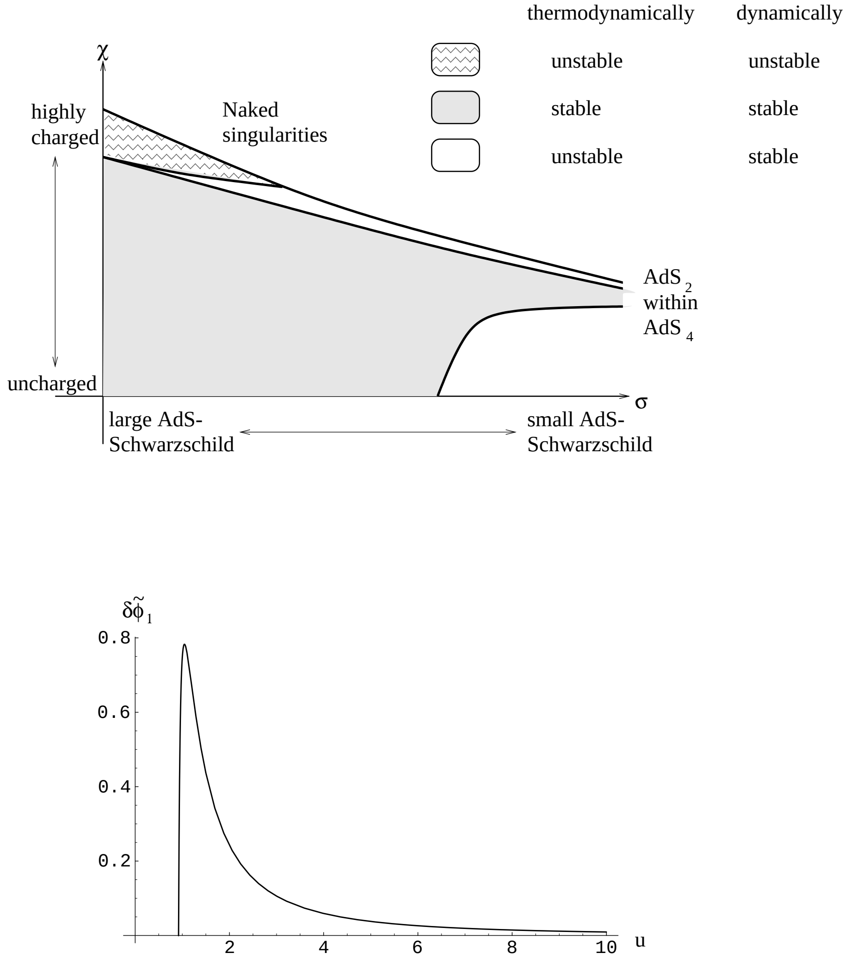

Figure 4: (a) A topologically correct representation of dynamical and thermodynamic stability in the whole χ-σ plane (but see the text regarding possible overlap of the two shaded regions). (b) A sample normalizable wave-function with negative ω^2: here $\sigma = 0.3$, $\chi = 0.96$, and $\tilde{\omega}^2 = -0.281$.

A qualitative summary of our numerical results is displayed in figure 4(a). An example of a normalizable wave-function with negative ω^2 is shown in figure 4(b). Some points to note are:

- The boundary of the region of dynamical stability comes from instability in the $\ell = 1$ mode. The $\ell = 0$ mode is projected out by charge conservation. Higher ℓ modes become unstable in the upper left part of the shaded triangle in figure 4(b). The boundaries of dynamical instability for different ℓ all come together at $\sigma = 0$.

- At $\sigma = 0$, thermodynamic stability is lost at $\chi = 1$, whereas dynamical instability sets in at $\chi = 1.007$. We believe that the 0.7% discrepancy is due to numerical error.

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- We have drawn the regions of dynamical instability and thermodynamic stability as disjoint in figure 4(a). In fact, our current numerics shows them overlapping by about 0.1% around $\sigma = 0.1$. We do not view this as significant because the numerical errors seem to be around 1%.

Finally, it is worth pointing out that the string theory program of computing black hole entropy via a microscopic state count in a field theory dual (see for example [46], or [47] for a review) has proved hard to extend past the boundaries of thermodynamic stability. For instance, we have a good understanding of the entropy of near-extremal D3-branes [48, 49], but not of small Schwarzschild black holes in AdS. It seems to us that this is no accident: most sensible field theories have log-convex partition functions, and this translates into hessian matrices $\mathbf{H}^S_{M,Q_A}$ which have no negative eigenvalues. Pushing past the boundary of thermodynamic stability in a field theory may be possible (particularly as one crosses a phase boundary and begins to nucleate the new phase), but doing so seems likely to produce dynamical instabilities in the lorentzian time-evolution. This point of view has indeed informed our entire investigation.

A dual field theory description of a small Schwarzschild black hole in AdS must involve thermodynamic instability but no dynamical instabilities. We believe that finite volume effects in the field theory are essential in this regard: if one imagines a Landau-Ginzburg effective description of the field theory, then derivative terms must restore stability to a system whose infrared tendencies are controlled by the thermodynamic instability. Various properties of small AdS-Schwarzschild black holes have been explored (see for example [50, 51]), but the basic problem of reconciling thermodynamic instability with dynamical stability in the presence of a field theory dual remains to be addressed.

5. Conclusions

A common conception of the Gregory-Laflamme instability is that a uniform solution to Einstein's equations (plus matter) competes with a non-uniform solution, and the non-uniform solution sometimes wins out entropically. In such a situation, the generic expectation is that there is a first order tunneling transition from the uniform to the non-uniform state, which may take place very slowly due to a large energetic barrier. In fact, the original papers [1, 2] focused mainly on demonstrating the existence of unstable modes in a linearized perturbation analysis of the uniform solution. The distinction is between global and local stability. At the level of classical gravity/field theory, the latter concept is more meaningful, because with quantum effects suppressed it is impossible to tunnel away from a locally stable solution. The aim of this paper and its shorter companion paper [17] has been to study local dynamical stability of black holes in anti-de Sitter space in relation to a particular notion of local thermodynamic stability, namely downward concavity of the entropy as a function of the other extensive variables. We reach two main conclusions:

1. In the limit of large black holes in AdS, dynamical and thermodynamic stability coincide. This conclusion is supported by numerical evidence. The small discrepancy between the observed onset of dynamical and thermodynamic instabilities is probably numerical error.

2. Dynamical instabilities persist for finite size black holes in AdS, down to horizon radii on the order of the AdS radius. The evidence is again only numerical, but we believe the final answer is correct and robust.

We regard point 1 as a partial verification of a rather more general conjecture, namely that black branes should have Gregory-Laflamme instabilities (in the local, dynamical sense of the original papers [1, 2]) precisely when thermodynamic stability is lost.

Point 2 is surprising because it is the first known example of a stationary black hole solution with a point-like singularity which exhibits a dynamical Gregory-Laflamme instability. Furthermore, it shows that no-hair theorems cannot always hold in anti-de Sitter space. Black branes which experience a Gregory-Laflamme instability are often supposed to split their horizons and fall into pieces. For this to happen to a horizon whose topology is $\mathbf{S}^2$ and which cloaks a point-like singularity would be truly novel: what could tear apart a point-like singularity?

Is Cosmic Censorship really threatened by our analysis?[12] It is too early to say. Using the heuristic method of calculating the most unstable eigenvector of the hessian of the entropy function, we have argued that adiabatic evolution of unstable black holes does not lead to nakedly singular solutions. However this does not bear directly on the question of whether the horizon should split as it is assumed to do in the evolution of unstable black branes. We leave open many questions as to the eventual fate of unstable black holes in AdS: Might they settle down to new non-uniform stationary solutions? Do regions of strong curvature form? Does the horizon split? Does AdS itself fragment through a classical process? We leave these issues for future work.

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[12]If asymptotically flat spacetimes are part of the hypothesis of Cosmic Censorship, as is often the case, then of course no demonstration in global anti-de Sitter space is relevant. We prefer a broader interpretation of Cosmic Censorship — loosely speaking, that no observer who follows a timelike trajectory which never runs into singularities can receive signals from a singularity.

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References

[1] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, *Phys. Rev. Lett.* **70** (1993) 2837 [hep-th/9301052].

[2] R. Gregory and R. Laflamme, *The instability of charged black strings and p-branes*, *Nucl. Phys.* **B 428** (1994) 399 [hep-th/9404071].

[3] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[4] C. Csaki, H. Ooguri, Y. Oz and J. Terning, *Glueball mass spectrum from supergravity*, *J. High Energy Phys.* **01** (1999) 017 [hep-th/9806021].

[5] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *large-N field theories, string theory and gravity*, *Phys. Rep.* **323** (2000) 183 [hep-th/9905111].

[6] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[7] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[8] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[9] V. Balasubramanian, P. Kraus and A.E. Lawrence, *Bulk vs. boundary dynamics in anti-de Sitter spacetime*, *Phys. Rev.* **D 59** (1999) 046003 [hep-th/9805171].

[10] S.S. Gubser, *Thermodynamics of spinning D3-branes*, *Nucl. Phys.* **B 551** (1999) 667 [hep-th/9810225].

[11] M. Cvetič and S.S. Gubser, *Phases of R-charged black holes, spinning branes and strongly coupled gauge theories*, *J. High Energy Phys.* **04** (1999) 024 [hep-th/9902195].

[12] M. Cvetič and S.S. Gubser, *Thermodynamic stability and phases of general spinning branes*, *J. High Energy Phys.* **07** (1999) 010 [hep-th/9903132].

[13] R.-G. Cai and K.-S. Soh, *Localization instability in the rotating D-branes*, *J. High Energy Phys.* **05** (1999) 025 [hep-th/9903023].

[14] T. Harmark and N.A. Obers, *Thermodynamics of spinning branes and their dual field theories*, *J. High Energy Phys.* **01** (2000) 008 [hep-th/9910036].

[15] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, *Phys. Rev.* **D 60** (1999) 064018 [hep-th/9902170].

JHEP08(2001)018

[16] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Holography, thermodynamics and fluctuations of charged AdS black holes*, *Phys. Rev.* **D 60** (1999) 104026 [hep-th/9904197].

[17] S.S. Gubser and I. Mitra, *Instability of charged black holes in anti-de sitter space*, hep-th/0009126.

[18] S.W. Hawking and H.S. Reall, *Charged and rotating AdS black holes and their CFT duals*, *Phys. Rev.* **D 61** (2000) 024014 [hep-th/9908109].

[19] S. Hawking, "Stability in ADS and Phase Transitions," talk at *Strings '99*, http://strings99.aei-potsdam.mpg.de/cgi-bin/viewit.cgi?speaker=Hawking

[20] T. Banks, M.R. Douglas, G.T. Horowitz and E. Martinec, *AdS dynamics from conformal field theory*, hep-th/9808016.

[21] S.W. Hawking and D.N. Page, *Thermodynamics of black holes in anti-de Sitter space*, *Comm. Math. Phys.* **87** (1983) 577.

[22] D.J. Gross, M.J. Perry and L.G. Yaffe, *Instability of flat space at finite temperature*, *Phys. Rev.* **D 25** (1982) 330.

[23] T. Prestidge, *Dynamic and thermodynamic stability and negative modes in Schwarzschild-anti-de Sitter*, *Phys. Rev.* **D 61** (2000) 084002 [hep-th/9907163].

[24] T. Prestidge, *Making $ense of the information loss paradox*, D.Phil. thesis, Cambridge University, January 2000.

[25] C.S. Peça and L. Lemos, *Thermodynamics of Reissner-Nordstrom anti-de Sitter black holes in the grand canonical ensemble*, *Phys. Rev.* **D 59** (1999) 124007 [gr-qc/9805004].

[26] C.S. Peça and L. Lemos, *Thermodynamics of toroidal black holes in anti-de Sitter spacetimes*, *J. Math. Phys.* **41** (2000) 4783 [gr-qc/9809029].

[27] R.H. Price, *Nonspherical perturbations of relativistic gravitational collapse. 1. scalar and gravitational perturbations*, *Phys. Rev.* **D 5** (1972) 2419.

[28] S.B. Giddings, J.A. Harvey, J.G. Polchinski, S.H. Shenker and A. Strominger, *Hairy black holes in string theory*, *Phys. Rev.* **D 50** (1994) 6422 [hep-th/9309152].

[29] D. Sahakian, to appear.

[30] M. Berkooz and M. Rozali, *Near hagedorn dynamics of NS fivebranes, or a new universality class of coiled strings*, *J. High Energy Phys.* **05** (2000) 040 [hep-th/0005047].

[31] T. Harmark and N.A. Obers, *Hagedorn behaviour of little string theory from string corrections to NS5-branes*, *Phys. Lett.* **B 485** (2000) 285 [hep-th/0005021].

[32] V. Kazakov, I. Kostov, and D. Kutasov, to appear.

[33] B. de Wit and H. Nicolai, *N=8 supergravity with local* SO(8)×SU(8) *invariance*, *Phys. Lett.* **B 108** (1982) 285.

[34] B. de Wit and H. Nicolai, $N = 8$ *supergravity*, *Nucl. Phys.* **B 208** (1982) 323.

[35] B. de Wit and H. Nicolai, *The consistency of the S^7 truncation in $d = 11$ supergravity*, *Nucl. Phys.* **B 281** (1987) 211.

[36] M.J. Duff and J.T. Liu, *Anti-de Sitter black holes in gauged $N = 8$ supergravity*, *Nucl. Phys.* **B 554** (1999) 237 [hep-th/9901149].

[37] J. Maldacena, J. Michelson and A. Strominger, *Anti-de Sitter fragmentation*, *J. High Energy Phys.* **02** (1999) 011 [hep-th/9812073].

[38] P. Kraus, F. Larsen and S.P. Trivedi, *The coulomb branch of gauge theory from rotating branes*, *J. High Energy Phys.* **03** (1999) 003 [hep-th/9811120].

[39] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Continuous distributions of D3-branes and gauged supergravity*, *J. High Energy Phys.* **07** (2000) 038 [hep-th/9906194].

[40] M. Cvetič, S.S. Gubser, H. Lu and C.N. Pope, *Symmetric potentials of gauged supergravities in diverse dimensions and coulomb branch of gauge theories*, *Phys. Rev.* **D 62** (2000) 086003 [hep-th/9909121].

[41] C. Csaki, Y. Oz, J. Russo and J. Terning, *Large-N QCD from rotating branes*, *Phys. Rev.* **D 59** (1999) 065012 [hep-th/9810186].

[42] E. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, *J. Math. Phys.* **3** (1962) 566.

[43] R. M. Wald, *General Relativity*, Chicago University Press, Chicago 1984.

[44] S.S. Gubser, *Absorption of photons and fermions by black holes in four dimensions*, *Phys. Rev.* **D 56** (1997) 7854 [hep-th/9706100].

[45] S.A. Teukolsky, *Perturbations of a rotating black hole. 1. Fundamental equations for gravitational electromagnetic and neutrino field perturbations*, *Astrophys. J.* **185** (1973) 635–647.

[46] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029].

[47] A.W. Peet, *Tasi lectures on black holes in string theory*, hep-th/0008241.

[48] S.S. Gubser, I.R. Klebanov and A.W. Peet, *Entropy and temperature of black 3-branes*, *Phys. Rev.* **D 54** (1996) 3915 [hep-th/9602135].

[49] A. Strominger, unpublished, 1996.

[50] G.T. Horowitz and V.E. Hubeny, *Quasinormal modes of AdS black holes and the approach to thermal equilibrium*, *Phys. Rev.* **D 62** (2000) 024027 [hep-th/9909056].

[51] G.T. Horowitz and V.E. Hubeny, *CFT description of small objects in AdS*, *J. High Energy Phys.* **10** (2000) 027 [hep-th/0009051].